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Noncommutative $D = 4$ gravity coupled to fermions

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Abstract: We present a noncommutative extension of Einstein-Hilbert gravity in the context of twist-deformed space-time, with a $\star$-product associated to a quite general triangular Drinfeld twist. In particular the $\star$-product can be chosen to be the usual Groenewald-Moyal product. The action is geometric, invariant under diffeomorphisms and centrally extended Lorentz $\star$-gauge transformations. In the commutative limit it reduces to ordinary gravity, with local Lorentz invariance and usual real vielbein. This we achieve by imposing a charge conjugation condition on the noncommutative vielbein. The theory is coupled to fermions, by adding the analog of the Dirac action in curved space. A noncommutative Majorana condition can be imposed, consistent with the $\star$-gauge transformations. Finally, we discuss the noncommutative version of the Mac-Dowell Mansouri action, quadratic in curvatures.

Keywords: Non-Commutative Geometry, Classical Theories of Gravity, Space-Time Symmetries

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Contents

1 Introduction

Field theories on noncommutative twisted spaces have been the object of active and recent research: they can be considered as field theories on ordinary spacetime where the product of fields is deformed into a twisted, noncommutative and associative $\star$-product. This product generates infinitely many derivatives on the fields and introduces a dimensionful noncommutativity parameter θ. Usually the first step is to take the classical theory and deform it by replacing ordinary products by $\star$-products. This is a way to deform a theory by introducing an infinite number of new interactions and higher derivative terms. Some noncommutative deformations of scalar field theories exhibit nontrivial symmetries [1] and provide new renormalizable models [2], others lead to new nontrivial integrable systems [3]. Noncommutative gauge theories have been intensively studied: they naturally arise under T-duality [4], and also describe a low energy sector of D-branes physics [5]. The renormalizability of these theories is still problematic. Some partial results can be found in ref.s [6].

Noncommutative gravity theories have been constructed in the past in the context of particular quantum groups [7] and more recently in the twisted noncommutative geometry setting [8, 9, 11]. In the second order formalism of ref. [9] the deformed theory is invariant under diffeomorphisms, but no gauge invariance on the tangent space (generalizing local Lorentz symmetry) is considered, and therefore coupling to fermions has not been discussed. In ref. [8] the noncommutative gravity action has a local $GL(2, C)$ invariance acting on tangent indices, but reduces in the commutative limit to gravity with a complex vielbein. Other attempts to formulate noncommutative deformations of gravity in the first order formalism can be found in [12].

In this paper, by using the tools of twisted differential geometry, we construct a geometric theory of noncommutative gravity. The Lagrangian is seen to be a globally defined 4-form, hence invariant under diffeomorphisms as well as $\star$-diffeomorphisms. Since these latter do not change the $\star$-product, they are a symmetry of the theory. The action is also invariant under a $GL(2, C)$ $\star$-gauge symmetry[1] that reduces to ordinary local Lorentz symmetry in the commutative limit. This we achieve by $\star$-extending the first order formalism of gravity coupled to Dirac fermions, formulated in a convenient index-free form. For the bosonic part our treatment does not differ much from the approach of Chamseddine. However we find a charge conjugation condition on the noncommutative vierbein field, consistent with the $\star$-gauge variations, that ensures the usual commutative limit. Thus we do not have to cope with an extra vierbein in the $\theta \to 0$ limit, as in [8]. The charge conjugation condition involves also the θ dependence of the fields. These we can imagine expanded in powers of θ, and in principle this picture introduces infinitely many fields, one for each power of θ. If we wish, we can use the Seiberg-Witten map to express all fields in terms of the classical one, thereby ending up with a finite number of fields.

We then discuss a noncommutative Majorana condition, that allows coupling of noncommutative gravity to Majorana fermions. The coupling of first order gravity to a Rarita-Schwinger fermion (gravitino), a noncommutative generalization of $D = 4$, $N = 1$ supergravity, is discussed in a companion paper [14].

[1] In this paper noncommutative gauge symmetries are called for short $\star$-gauge symmetries and should not be confused with the twisted gauge symmetries discussed in [13].

The quantum treatment of the resulting higher-derivative theory is still virgin territory: in this case, at least, we cannot do worse than in the commutative limit, where the theory is not renormalizable and not finite. We do not expect the infinitely many derivatives to conspire with the infinities present in the commutative theory, and miraculously cancel its divergences. In fact the Drinfeld twist corresponds to a braiding matrix with unit square, and this leads to a differential geometry with usual derivatives, i.e. infinitesimal difference operators. On the other hand, we know that on quasi-triangular quantum groups derivatives become finite difference operators: then one can indeed expect that theories constructed with the tools of quasi-triangular differential geometry may be regulated by noncommutativity. In this perspective the present study of twisted gravity may be seen as a preparatory step.

The paper proceeds as follows. In section 2 we recast the usual first order gravity coupled to a fermion field in an index-free form, convenient for its noncommutative extension. Section 3 recalls some basic tools of twisted noncommutative geometry, and in section 4 and 5 we present the action and the invariances of the noncommutative theory. Section 6 deals with the noncommutative version of MacDowell-Mansouri quadratic gravity. An appendix on gamma matrices summarizes our conventions.

2 First order gravity coupled to fermions

2.1 Action

The usual action of first-order gravity coupled to fermions can be recast in an index-free form, convenient for generalization to the non-commutative case:

$$S = \int Tr\left(iR \wedge V \wedge V\gamma_5 - \left[(D\psi)\bar{\psi} - \psi D\bar{\psi}\right] \wedge V \wedge V \wedge V\gamma_5\right) \tag{2.1}$$

The fundamental fields are the 1-forms Ω (spin connection), V (vielbein) and the fermionic 0-form ψ (spin 1/2 field). The curvature 2-form R and the exterior covariant derivative on ψ are defined by

$$R = d\Omega - \Omega \wedge \Omega, \qquad D\psi = d\psi - \Omega\psi \tag{2.2}$$

with

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab}, \qquad V = V^a\gamma_a \tag{2.3}$$

and thus are 4×4 matrices in the spinor representation. See appendix A for $D = 4$ gamma matrix conventions and useful relations. The Dirac conjugate is defined as usual: $\bar{\psi} = \psi^\dagger\gamma_0$. Then also $(D\psi)\bar{\psi}$, $\psi D\bar{\psi}$ are matrices in the spinor representation, and the trace Tr is taken on this representation. Using the $D = 4$ gamma matrix identities:

$$\gamma_{abc} = i\varepsilon_{abcd}\gamma^d\gamma_5, \qquad Tr(\gamma_{ab}\gamma_c\gamma_d\gamma_5) = -4i\varepsilon_{abcd} \tag{2.4}$$

leads to the usual action:

$$S = \int R^{ab} \wedge V^c \wedge V^d\varepsilon_{abcd} + i\left[\bar{\psi}\gamma^a D\psi - (D\bar{\psi})\gamma^a\psi\right] \wedge V^b \wedge V^c \wedge V^d\varepsilon_{abcd} \tag{2.5}$$

with

$$R \equiv \frac{1}{4}R^{ab}\gamma_{ab}, \qquad R^{ab} = d\omega^{ab} - \omega^a_{\ c} \wedge \omega^{cb} \tag{2.6}$$

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2.2 Invariances

The action is invariant under local diffeomorphisms (it is the integral of a 4-form on a 4-manifold) and under the local Lorentz rotations:

$$\delta_\epsilon V = -[V, \epsilon], \qquad \delta_\epsilon \Omega = d\epsilon - [\Omega, \epsilon], \qquad \delta_\epsilon \psi = \epsilon \psi, \qquad \delta_\epsilon \bar{\psi} = -\bar{\psi}\epsilon \tag{2.7}$$

with

$$\epsilon = \frac{1}{4}\epsilon^{ab}\gamma_{ab} \tag{2.8}$$

The invariance can be directly checked on the action (2.1) noting that

$$\delta_\epsilon R = -[R, \epsilon], \quad \delta_\epsilon D\psi = \epsilon D\psi, \quad \delta_\epsilon((D\psi)\bar{\psi}) = -[(D\psi)\bar{\psi}, \epsilon], \quad \delta_\epsilon(\psi D\bar{\psi}) = -[\psi D\bar{\psi}, \epsilon], \tag{2.9}$$

using the cyclicity of the trace Tr (on spinor indices) and the fact that ϵ commutes with γ_5. The Lorentz rotations close on the Lie algebra:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = -\delta_{[\epsilon_1, \epsilon_2]} \tag{2.10}$$

2.3 Hermiticity and charge conjugation

Since the vielbein V^a and the spin connection ω^{ab} are real fields, the following conditions hold:

$$\gamma_0 V \gamma_0 = V^\dagger, \qquad\qquad -\gamma_0 \Omega \gamma_0 = \Omega^\dagger, \tag{2.11}$$

$$\gamma_0 \left[(D\psi)\bar{\psi}\right] \gamma_0 = \left[\psi D\bar{\psi}\right]^\dagger, \qquad\qquad \gamma_0 \left[\psi D\bar{\psi}\right] \gamma_0 = \left[(D\psi)\bar{\psi}\right]^\dagger \tag{2.12}$$

and can be used to check that the action (2.1) is real.

Moreover, if C is the $D = 4$ charge conjugation matrix (antisymmetric and squaring to -1), we have

$$CVC = V^T, \qquad C\Omega C = \Omega^T \tag{2.13}$$

since the matrices $C\gamma_a$ and $C\gamma_{ab}$ are symmetric.

Similar relations hold for the gauge parameter $\epsilon = (1/4)\varepsilon^{ab}\gamma_{ab}$:

$$-\gamma_0 \epsilon \gamma_0 = \epsilon^\dagger, \qquad C\epsilon C = \epsilon^T \tag{2.14}$$

ε^{ab} being real.

The charge conjugation of fermions:

$$\psi^C \equiv C(\bar{\psi})^T \tag{2.15}$$

can be extended to the bosonic fields V, Ω:

$$V^C \equiv -CV^TC, \qquad \Omega^C \equiv C\Omega^TC \tag{2.16}$$

Then the relations (2.13) can be written as:

$$V^C = -V, \qquad \Omega^C = \Omega \tag{2.17}$$

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and are the analogues of the Majorana condition for the fermions:

$$\psi^C = \psi \quad \rightarrow \quad \bar{\psi} = \psi^T C \tag{2.18}$$

Note also that

$$(V\psi)^C = V^C \psi^C \tag{2.19}$$

In particular, if ψ is a Majorana fermion, $V\psi$ is anti-Majorana. So far we have been treating ψ as a Dirac fermion, and therefore reality of the action requires both terms in square brackets in the action (2.1) or (2.5). If ψ is Majorana, the two terms give the same contribution, and only one of them is necessary.

2.4 Field equations

Using the cyclicity of Tr in (2.1), the variation of V , Ω and $\bar{\psi}$ yield respectively the Einstein equation, the torsion equation and the (massless) Dirac equation in index-free form:

$$Tr\Big(\gamma_a\gamma_5[iV \wedge R + iR \wedge V - X \wedge V \wedge V - V \wedge X \wedge V - V \wedge V \wedge X]\Big) = 0,$$

$$Tr\Big(\gamma_{ab}[iT \wedge V - iV \wedge T + \psi\bar{\psi}V \wedge V \wedge V - V \wedge V \wedge V\psi\bar{\psi}]\Big) = 0 \tag{2.20}$$

$$V \wedge V \wedge V \wedge D\psi - (T \wedge V \wedge V - V \wedge T \wedge V + V \wedge V \wedge T)\psi = 0 \tag{2.21}$$

with

$$X \equiv (D\psi)\bar{\psi} - \psi D\bar{\psi} \tag{2.22}$$

and where the torsion $T = T^a\gamma_a$ is given by:

$$T \equiv dV - \Omega \wedge V - V \wedge \Omega \tag{2.23}$$

The torsion equation can be solved, and yields the known result:

$$T^a = 6i\ \bar{\psi}\gamma_b\psi\ V^b \wedge V^a \tag{2.24}$$

The Dirac equation (2.21) contains an extra term proportional to the torsion: this is due to requiring a real action for gravity coupled to Dirac fermions. If one uses the (complex) Dirac action

$$S_{\text{Dirac}} = -\int Tr[(D\psi)\bar{\psi} \wedge V \wedge V \wedge V\gamma_5] \tag{2.25}$$

the torsion terms in the Dirac equation (2.21) are not present.

3 Twist differential geometry: some tools

The noncommutative deformation of the gravity theories we construct in the next sections relies on the existence (in the deformation quantization context, see for ex [15]) of an associative $\star$-product between functions and more generally an associative $\wedge_\star$ exterior product between forms, satisfying the following properties:

- Compatibility with the undeformed exterior differential:

$$d(\tau \wedge_\star \tau') = d(\tau) \wedge_\star \tau' + (-1)^{deg(\tau)}\tau \wedge_\star d\tau' \tag{3.1}$$

- Compatibility with the undeformed integral (graded cyclicity property):

$$\int \tau \wedge_\star \tau' = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge_\star \tau \tag{3.2}$$

with $deg(\tau)+deg(\tau')$ =D=dimension of the spacetime manifold, and where here τ and τ' have compact support (otherwise stated we require (3.2) to hold up to boundary terms).

- Compatibility with the undeformed complex conjugation:

$$(\tau \wedge_\star \tau')^* = (-1)^{deg(\tau)deg(\tau')} \tau'^* \wedge_\star \tau^* \tag{3.3}$$

We describe here a (quite wide) class of twists whose $\star$-products have all these properties. In this way we have constructed a wide class of noncommutative deformations of gravity theories. Of course as a particular case we have the Groenewold-Moyal $\star$-product

$$f \star g = \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} f \otimes g\}, \tag{3.4}$$

where the map μ is the usual pointwise multiplication: $\mu(f\otimes g) = fg$, and $\theta^{\rho\sigma}$ is a constant antisymmetric matrix.

3.1 Twist

Let Ξ be the linear space of smooth vector fields on a smooth manifold M, and $U\Xi$ its universal enveloping algebra. A twist $\mathcal{F} \in U\Xi \otimes U\Xi$ defines the associative twisted product

$$f \star g = \mu\{\mathcal{F}^{-1} f \otimes g\} \tag{3.5}$$

where the map μ is the usual pointwise multiplication: $\mu(f\otimes g) = fg$. The product associativity relies on the defining properties of the twist [9, 15, 16]. Using the standard notation

$$\mathcal{F} \equiv \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha, \qquad \mathcal{F}^{-1} \equiv \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha \tag{3.6}$$

(sum over α understood) where $\mathrm{f}^\alpha, \mathrm{f}_\alpha, \bar{\mathrm{f}}^\alpha, \bar{\mathrm{f}}_\alpha$ are elements of $U\Xi$, the $\star$-product is expressed in terms of ordinary products as:

$$f \star g = \bar{\mathrm{f}}^\alpha(f)\bar{\mathrm{f}}_\alpha(g) \tag{3.7}$$

Many explicit examples of twist are provided by the so-called abelian twists:

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a\otimes X_b} \tag{3.8}$$

where $\{X_a\}$ is a set of mutually commuting vector fields globally defined on the manifold,[2] and θ^{ab} is a constant antisymmetric matrix. The corresponding $\star$-product is in general

[2] We actually need only the twist $\mathcal{F}$ to be globally defined, not necessarily the single vector fields X_a. An explicit example of this latter kind is given by the twist (3.8), that in an open neighbourhood with coordinates t, x, y, z is defined by the commuting vector fields $X_1 = f(x,z)\frac{\partial}{\partial x}$, $X_2 = h(y,z)\frac{\partial}{\partial y}$, where $f(x,z)$ is a function of only the x and z variables and has compact support, and similarly $h(y,z)$. This twist is globally defined on the whole manifold by requiring it to be the identity $1\otimes 1$ outside the $\{x^a\}$ coordinate neighbourhood. The corresponding $\star$-product, defined on the whole spacetime manifold, is noncommutative only inside this neighbourhood.

position dependent because the vector fields X_a are in general x-dependent. In the special case that there exists a global coordinate system on the manifold we can consider the vector fields $X_a = \frac{\partial}{\partial x^a}$. In this instance we have the Moyal twist, cf. (3.4):

$$\mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma} \tag{3.9}$$

3.2 Deformed exterior product

The deformed exterior product between forms is defined as

$$\tau \wedge_\star \tau' \equiv \bar{\mathrm{f}}^\alpha(\tau) \wedge \bar{\mathrm{f}}_\alpha(\tau') \tag{3.10}$$

where $\bar{\mathrm{f}}^\alpha$ and $\bar{\mathrm{f}}_\alpha$ act on forms via the Lie derivatives $\mathcal{L}_{\bar{\mathrm{f}}^\alpha}$, $\mathcal{L}_{\bar{\mathrm{f}}_\alpha}$ (Lie derivatives along products $uv\cdots$ of elements of Ξ are defined simply by $\mathcal{L}_{uv\cdots} \equiv \mathcal{L}_u\mathcal{L}_v\cdots$). This product is associative, and in particular satisfies:

$$\tau \wedge_\star h \star \tau' = \tau \star h \wedge_\star \tau', \quad h \star (\tau \wedge_\star \tau') = (h \star \tau) \wedge_\star \tau', \quad (\tau \wedge_\star \tau') \star h = \tau \wedge_\star (\tau' \star h) \tag{3.11}$$

where h is a 0-form, i.e. a function belonging to $Fun(M)$, the $\star$-product between functions and one-forms being just a particular case of (3.10):

$$h \star \tau = \bar{\mathrm{f}}^\alpha(h)\bar{\mathrm{f}}_\alpha(\tau), \qquad \tau \star h = \bar{\mathrm{f}}^\alpha(\tau)\bar{\mathrm{f}}_\alpha(h) \tag{3.12}$$

3.3 Exterior derivative

The exterior derivative satisfies the usual (graded) Leibniz rule, since it commutes with the Lie derivative:

$$d(f \star g) = df \star g + f \star dg \tag{3.13}$$

$$d(\tau \wedge_\star \tau') = d\tau \wedge_\star \tau' + (-1)^{deg(\tau)}\, \tau \wedge_\star d\tau' \tag{3.14}$$

3.4 Integration: graded cyclicity

If we consider an abelian twist (3.8) given by globally defined commuting vector fields X_a, then the usual integral is cyclic under the $\star$-exterior products of forms, i.e., up to boundary terms,

$$\int \tau \wedge_\star \tau' = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge_\star \tau \tag{3.15}$$

with $deg(\tau) + deg(\tau')$ =D=dimension of the spacetime manifold. In fact we have

$$\int \tau \wedge_\star \tau' = \int \tau \wedge \tau' = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge \tau = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge_\star \tau \tag{3.16}$$

For example at first order in θ,

$$\int \tau \wedge_\star \tau' = \int \tau \wedge \tau' - \frac{i}{2}\theta^{ab} \int \mathcal{L}_{X_a}(\tau \wedge \mathcal{L}_{X_b}\tau') = \int \tau \wedge \tau' - \frac{i}{2}\theta^{ab} \int d i_{X_a}(\tau \wedge \mathcal{L}_{X_b}\tau') \tag{3.17}$$

where we used the Cartan formula $\mathcal{L}_{X_a} = di_{X_a} + i_{X_a}d$. More generally if the twist $\mathcal{F}$ satisfies the condition $S(\bar{\mathrm{f}}^\alpha)\bar{\mathrm{f}}_\alpha = 1$, where the antipode S is defined on vector fields as $S(v) = -v$ and is extended to the whole universal enveloping algebra $U\Xi$ linearly and antimultiplicatively, $S(uv) = S(v)S(u)$, then a similar argument proves the graded cyclicity of the integral.[3].

[3]Proof: we use Sweedler's coproduct notation $\Delta(\zeta) = \zeta_1 \otimes \zeta_2$, where $\zeta \in U\Xi$ and the coproduct map $\Delta : U\Xi \to U\Xi \otimes U\Xi$ is defined on vector fields $u \in \Xi$ by $\Delta(u) = u \otimes 1 + 1 \otimes u$ and is extended linearly

3.5 Complex conjugation

If we choose real fields X_a in the definition of the twist (3.8), it is immediate to verify that:

$$(f \star g)^* = g^* \star f^* \tag{3.18}$$
$$(\tau \wedge_\star \tau')^* = (-1)^{deg(\tau)deg(\tau')} \tau'^* \wedge_\star \tau^* \tag{3.19}$$

since sending i into $-i$ in the twist (3.9) amounts to send θ^{ab} into $-\theta^{ab} = \theta^{ba}$, i.e. to exchange the order of the factors in the $\star$-product. More in general we can consider twists $\mathcal{F}$ that satisfy the reality condition (cf. section 8 in [9]) $\bar{\mathrm{f}}^{\alpha *} \otimes \bar{\mathrm{f}}_\alpha{}^* = S(\bar{\mathrm{f}}_\alpha) \otimes S(\bar{\mathrm{f}}^\alpha)$. The $\star$-products associated to these twists satisfy properties (3.18), (3.19).

4 Noncommutative gravity coupled to fermions

4.1 Action and symmetries

Here we generalize section 2 to the noncommutative case, mostly by replacing exterior products by deformed exterior products. Thus the action becomes:

$$S = \int Tr \left(iR \wedge_\star V \wedge_\star V \gamma_5 - [(D\psi) \star \bar{\psi} - \psi \star D\bar{\psi}] \wedge_\star V \wedge_\star V \wedge_\star V \gamma_5 \right) \tag{4.1}$$

with

$$R = d\Omega - \Omega \wedge_\star \Omega, \qquad D\psi = d\psi - \Omega \star \psi \tag{4.2}$$

Almost all formulae in section 2 continue to hold, with $\star$-products and $\star$-exterior products. However, the expansion of the fundamental fields on the Dirac basis of gamma matrices must now include new contributions:

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} + i\omega 1 + \tilde{\omega}\gamma_5, \qquad V = V^a\gamma_a + \tilde{V}^a\gamma_a\gamma_5 \tag{4.3}$$

Similarly for the curvature:

$$R = \frac{1}{4}R^{ab}\gamma_{ab} + ir1 + \tilde{r}\gamma_5 \tag{4.4}$$

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and multiplicatively to all $U\Xi$ (cf. [10]). In the notation $\zeta_1 \otimes \zeta_2$ a sum is understood. The coproduct Δ encodes the Leibniz rule property: for example $\zeta(\tau \wedge \tilde{\tau}) = \zeta_1(\tau) \wedge \zeta_2(\tilde{\tau})$. The coproduct is coassociative: $(\zeta_1)_1 \otimes (\zeta_1)_2 \otimes \zeta_2 = \zeta_1 \otimes (\zeta_2)_1 \otimes (\zeta_2)_2$ and it is standard to denote this element simply by $\zeta_1 \otimes \zeta_2 \otimes \zeta_3$. Compatibility between the coproduct, the antipode and the product implies $\zeta_1 S(\zeta_2)(\tilde{\tau}) = \zeta(1)\, \tilde{\tau}$ (cf. [10]). Then we also have $\zeta_1(\tau) \wedge \zeta_2 S(\zeta_3)(\tilde{\tau}) = \zeta_1(\tau) \wedge \zeta_2(1)\tilde{\tau} = \zeta_1(\tau)\zeta_2(1) \wedge \tilde{\tau} = \zeta(\tau \cdot 1) \wedge \tilde{\tau} = \zeta(\tau) \wedge \tilde{\tau}$. We now apply this formula in the case $\zeta = \bar{\mathrm{f}}^\alpha$, and compute

$$\begin{aligned}\tau \wedge_\star \tau' &= \bar{\mathrm{f}}^\alpha(\tau) \wedge \bar{\mathrm{f}}_\alpha(\tau') = \bar{\mathrm{f}}^\alpha_1(\tau) \wedge \bar{\mathrm{f}}^\alpha_2 S(\bar{\mathrm{f}}^\alpha_3)\bar{\mathrm{f}}_\alpha(\tau') = \bar{\mathrm{f}}^\alpha_1(\tau \wedge S(\bar{\mathrm{f}}^\alpha_2)\bar{\mathrm{f}}_\alpha(\tau'))\\ &= \tau \wedge S(\bar{\mathrm{f}}^\alpha)\bar{\mathrm{f}}_\alpha(\tau') + \bar{\mathrm{f}}^{\alpha'}{}_1(\tau \wedge S(\bar{\mathrm{f}}^{\alpha'}{}_2)\bar{\mathrm{f}}_\alpha(\tau')) = \tau \wedge \tau' + \text{total derivative}\end{aligned}$$

where in the first line we have used the definition $\bar{\mathrm{f}}^\alpha_1 \otimes \bar{\mathrm{f}}^\alpha_2 \otimes \bar{\mathrm{f}}^\alpha_3 = \Delta(\bar{\mathrm{f}}^\alpha_1) \otimes \bar{\mathrm{f}}^\alpha_2$, and in the second line we observed that the coproduct $\Delta(\bar{\mathrm{f}}^\alpha) = \bar{\mathrm{f}}^\alpha_1 \otimes \bar{\mathrm{f}}^\alpha_2$ contains the term $1 \otimes \bar{\mathrm{f}}^\alpha$ (that is obtained by considering for each vector field entering $\bar{\mathrm{f}}^\alpha$ only the term $1 \otimes u$ of the coproduct rule $\Delta(u) = u \otimes 1 + 1 \otimes u$) plus other remaining terms that we denoted $\bar{\mathrm{f}}^{\alpha'}{}_1 \otimes \bar{\mathrm{f}}^{\alpha'}{}_2$. Now by construction each $\bar{\mathrm{f}}^{\alpha'}{}_1$ contains at least one vector field, if we assume that the twist satisfies $\mathcal{F}^{-1} = 1 \otimes 1 + \cdots$, where $\cdots$ denotes sums of (products of vector fields) $\otimes$ (products of vector fields). Since vector fields act via the Lie derivative, the Cartan formula $\mathcal{L}_u = i_u d + d i_u$ implies that $\bar{\mathrm{f}}^{\alpha'}{}_1(\tau \wedge S(\bar{\mathrm{f}}^{\alpha'}{}_2)\bar{\mathrm{f}}_\alpha(\tau'))$ is a total derivative, the Lie derivative acting on a form of highest degree (top form) so that its $i_u d$ part vanishes.

and for the gauge parameter:

$$\epsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} + i\varepsilon 1 + \tilde{\varepsilon}\gamma_5 \tag{4.5}$$

Indeed now the $\star$-gauge variations read:

$$\delta_\epsilon V = -V \star \epsilon + \epsilon \star V, \quad \delta_\epsilon \Omega = d\epsilon - \Omega \star \epsilon + \epsilon \star \Omega, \quad \delta_\epsilon \psi = \epsilon \star \psi, \quad \delta_\epsilon \bar{\psi} = -\bar{\psi} \star \epsilon \tag{4.6}$$

and in the variations for V and Ω also anticommutators of gamma matrices appear, due to the noncommutativity of the $\star$-product. Since for example the anticommutator $\{\gamma_{ab}, \gamma_{cd}\}$ contains 1 and γ_5, we see that the corresponding fields must be included in the expansion of Ω. Similarly, V must contain a $\gamma_a\gamma_5$ term due to $\{\gamma_{ab}, \gamma_c\}$. Finally, the composition law for gauge parameters becomes:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_2 \star \epsilon_1 - \epsilon_1 \star \epsilon_2} \tag{4.7}$$

so that ϵ must contain the 1 and γ_5 terms, since they appear in the composite parameter $\epsilon_2 \star \epsilon_1 - \epsilon_1 \star \epsilon_2$.

The invariance of the noncommutative action (4.1) under the $\star$-variations is demonstrated in exactly the same way as for the commutative case, noting that

$$\delta_\epsilon R = -R \star \epsilon + \epsilon \star R, \quad \delta_\epsilon D\psi = \epsilon \star D\psi, \quad \delta_\epsilon((D\psi) \star \bar{\psi}) = -(D\psi) \star \bar{\psi} \star \epsilon + \epsilon \star (D\psi) \star \bar{\psi} \tag{4.8}$$

etc., and using now, besides the cyclicity of the trace Tr and the fact that ϵ still commutes with γ_5, also the graded cyclicity of the integral.

The local $\star$-symmetry satisfies the Lie algebra of $GL(2, C)$, and centrally extends the SO$(1, 3)$ Lie algebra of the commutative theory.

Finally, the $\star$-action (4.1) is invariant under diffeomorphisms generated by the Lie derivative, in the sense that

$$\int \mathcal{L}_v(4\text{−form}) = \int (i_v d + d i_v)(4\text{−form}) = \int d(i_v(4\text{−form})) = \textit{boundary term} \tag{4.9}$$

since $d(4\text{−form}) = 0$ on a 4-dimensional manifold.[4]

We have constructed a geometric lagrangian where the fields are exterior forms and the $\star$-product is given by the Lie derivative action of the twist on forms. The twist $\mathcal{F}$ in general is not invariant under the diffeomorphism $\mathcal{L}_v$. However we can consider the $\star$-diffeomorphisms of ref. [9] (see also [15], section 8.2.4), generated by the $\star$-Lie derivative. This latter acts trivially on the twist $\mathcal{F}$ but satisfies a deformed Leibniz rule. $\star$-Lie derivatives generate infinitesimal noncommutative diffeomorphisms and leave invariant the action and the twist. They are noncommutative symmetries of our action.

[4] In order to show that the integrand is a globally defined 4-form we need to assume that the vielbein one-form V^a is globally defined (and therefore that the manifold is parallelizable), the twisted exterior product being globally defined (because the twist is globally defined). If this is the case, then due to the local $GL(2, C)$ $\star$-invariance the action is independent of the vielbein used. On the other hand, if the vielbein V^a is only locally defined in open coverings of the manifold, then we cannot construct a global 4-form, since the local $GL(2, C)$ $\star$-invariance holds only under integration.

Finally in our geometric action no coordinate indices μ, ν appear, and this implies invariance of the action under (undeformed) general coordinate transformations.[5] Otherwise stated every contravariant tensor index $^\mu$ is contracted with the corresponding covariant tensor index $_\mu$, for example $X_a = X_a^\mu \partial_\mu$ and $V^a = V_\mu^a dx^\mu$.

4.2 Field equations

Using the cyclicity of Tr and the graded cyclicity of the integral in (4.1), the variation of V, Ω and $\bar{\psi}$ yield respectively the noncommutative Einstein equation, torsion equation and Dirac equation in index-free form:

$$\begin{aligned}
Tr[\Gamma_{a,a5}(iV \wedge_\star R + iR \wedge_\star V - X \wedge_\star V \wedge_\star V - V \wedge_\star X \wedge_\star V - V \wedge_\star V \wedge_\star X)] &= 0 \\
Tr[\Gamma_{ab,1,5}(iT \wedge_\star V - iV \wedge_\star T + \psi \star \bar{\psi} \star V \wedge_\star V \wedge_\star V - V \wedge_\star V \wedge_\star V \star \psi \star \bar{\psi})] &= 0 \\
V \wedge_\star V \wedge_\star V \wedge_\star D\psi - (T \wedge_\star V \wedge_\star V - V \wedge_\star T \wedge_\star V + V \wedge_\star V \wedge_\star T) \star \psi &= 0
\end{aligned} \tag{4.10}$$

where $\Gamma_{a,a5}$ indicates γ_a and $\gamma_a \gamma_5$ (thus there are two distinct equations) and likewise for $\Gamma_{ab,1,5}$ (three equations corresponding to γ_{ab}, 1 and γ_5). The noncommutative torsion two-form is defined by:

$$T \equiv T^a \gamma_a + \tilde{T}^a \gamma_a \gamma_5 \equiv dV - \Omega \wedge_\star V - V \wedge_\star \Omega \tag{4.11}$$

The torsion equation (4.10) can be written as:

$$[iT \wedge_\star V - iV \wedge_\star T + \psi \star \bar{\psi} \star V \wedge_\star V \wedge_\star V - V \wedge_\star V \wedge_\star V \star \psi \star \bar{\psi}, \gamma_5]_+ = 0 \tag{4.12}$$

Indeed the anticommutator with γ_5 selects the γ_{ab}, 1 and γ_5 components. This equation can be solved for the torsion:

$$T = \frac{i}{2}[\psi \star \bar{\psi} \star V \wedge_\star V + V \wedge_\star \psi \star \bar{\psi} \star V + V \wedge_\star V \star \psi \star \bar{\psi}, \gamma_5]\gamma_5 \tag{4.13}$$

as can be verified by substitution into (4.12).

4.3 θ - dependent fields

We can rewrite the Moyal twist as:

$$\mathcal{F}^{-1} = e^{\frac{i}{2}\theta \Theta^{\rho\sigma} \partial_\rho \otimes \partial_\sigma} \tag{4.14}$$

where θ is a dimensionful parameter (so that $\Theta^{\rho\sigma}$ is a numerical matrix). In the spirit of the Seiberg-Witten map [5], the fields and the gauge parameter can be considered functions of x and θ. Expanding a field ϕ in powers of θ:

$$\phi_\theta(x) = \phi_0(x) + \theta\phi_1(x) + \theta^2\phi_2(x) + \ldots, \quad \varepsilon_\theta(x) = \varepsilon_0(x) + \theta\varepsilon_1(x) + \theta^2\varepsilon_2(x) + \ldots \tag{4.15}$$

introduces an infinite tower of x - dependent fields: a finite number of them enters in the action (4.1) at each given order in θ. At 0-th order only the classical fields $\phi_0(x)$ contribute.

[5] General coordinate transformations are diffeomorphisms of an open coordinate neighbourhood of the manifold, not of the whole manifold.

The gauge variations of all ϕ_i are deduced by expanding the $\star$-gauge transformations in (4.6) in powers of θ. Clearly the classical fields ϕ_0 transform with the classical gauge variations δ^0_ϵ.

If one feels uncomfortable with these new fields ϕ_i, the Seiberg-Witten map can be used to relate the higher-order fields to the classical ones in a way consistent with the $\star$ -gauge transformations δ_ϵ:

$$\delta_\epsilon \phi(\phi_0) = \phi(\delta^0_\epsilon \phi_0) \tag{4.16}$$

so that the $\star$-deformed theory will contain only the ϕ_0 fields [5, 17].

All the fields V^a, $\tilde{V}^a$, ω^{ab}, ω, and $\tilde{\omega}$ contained in the action (4.1) are then θ-expanded, and the 0-th order action contains their $\theta \to 0$ limit.

4.4 Hermiticity and charge conjugation

Hermiticity conditions can be imposed on V, Ω and the gauge parameter ϵ:

$$\gamma_0 V \gamma_0 = V^\dagger, \qquad -\gamma_0 \Omega \gamma_0 = \Omega^\dagger, \qquad -\gamma_0 \epsilon \gamma_0 = \epsilon^\dagger \tag{4.17}$$

Moreover it is easy to verify the analogues of conditions (2.12):

$$\gamma_0 [(D\psi) \star \bar{\psi}] \gamma_0 = [\psi \star D\bar{\psi}]^\dagger, \qquad \gamma_0 [\psi \star D\bar{\psi}] \gamma_0 = [D\psi \star \bar{\psi}]^\dagger \tag{4.18}$$

These hermiticity conditions are consistent with the gauge variations, as in the commutative case, and can be used to check that the action (4.1) is real. On the component fields V^a, $\tilde{V}^a$, ω^{ab}, ω, and $\tilde{\omega}$, and on the component gauge parameters ε^{ab}, ε, and $\tilde{\varepsilon}$ the hermiticity conditions (4.17) imply that they are real fields.

The charge conjugation relations (2.13), however, cannot be exported to the noncommutative case as they are. Indeed they would imply the vanishing of the component fields $\tilde{V}^a$, ω, and $\tilde{\omega}$ (whose presence is necessary in the noncommutative case) and anyhow would not be consistent with the $\star$-gauge variations.

An essential modification is needed, and makes use of the θ dependence of the noncommutative fields:

$$C V_\theta(x) C = V_{-\theta}(x)^T, \qquad C \Omega_\theta(x) C = \Omega_{-\theta}(x)^T, \qquad C \varepsilon_\theta(x) C = \varepsilon_{-\theta}(x)^T \tag{4.19}$$

These conditions can be checked to be consistent with the $\star$-gauge transformations. For example $C V_\theta(x)^T C$ can be shown to transform in the same way as $V_{-\theta}(x)$:

$$\begin{aligned} \delta_\epsilon (C V_\theta^T C) &= C(\delta_\epsilon V_\theta)^T C = C(-\epsilon_\theta^T \star_{-\theta} V_\theta^T + V_\theta^T \star_{-\theta} \epsilon_\theta^T) C = \\ &= \epsilon_{-\theta} \star_{-\theta} V_{-\theta} - V_{-\theta} \star_{-\theta} \epsilon_{-\theta} = \delta_\epsilon V_{-\theta} \end{aligned} \tag{4.20}$$

where we have used $C^2 = -1$ and the fact that the transposition of a $\star$-product of matrix-valued fields interchanges the order of the matrices but not of the $\star$-multiplied fields. To interchange both it is necessary to use the "reflected" $\star_{-\theta}$ product obtained by changing the sign of θ, since

$$f \star_\theta g = g \star_{-\theta} f \tag{4.21}$$

for any two functions f, g.

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For the component fields and gauge parameters the charge conjugation conditions imply:

$$V^a_\theta = V^a_{-\theta}, \qquad \omega^{ab}_\theta = \omega^{ab}_{-\theta} \tag{4.22}$$
$$\tilde V^a_\theta = -\tilde V^a_{-\theta}, \qquad \omega_\theta = -\omega_{-\theta}, \qquad \tilde\omega_\theta = -\tilde\omega_{-\theta}, \tag{4.23}$$

Similarly for the gauge parameters:

$$\varepsilon^{ab}_\theta = \varepsilon^{ab}_{-\theta} \tag{4.24}$$
$$\varepsilon_\theta = -\varepsilon_{-\theta}, \qquad \tilde\varepsilon_\theta = -\tilde\varepsilon_{-\theta} \tag{4.25}$$

Finally, let us consider the charge conjugate spinor:

$$\psi^C \equiv C(\bar\psi)^T \tag{4.26}$$

It transforms under $\star$-gauge variations as:

$$\delta_\epsilon \psi^C = C(\delta_\epsilon \bar\psi)^T = C(-\bar\psi \star \epsilon)^T = C(-\epsilon^T \star_{-\theta} \psi^*) = C\epsilon^T C \star_{-\theta} C\psi^* = \epsilon_{-\theta} \star_{-\theta} \psi^C \tag{4.27}$$

i.e. it transforms in the same way as $\psi_{-\theta}$. Then we can impose the noncommutative Majorana condition:

$$\psi^C_\theta = \psi_{-\theta} \quad \Rightarrow \quad \psi^\dagger_\theta \gamma_0 = \psi^T_{-\theta} C \tag{4.28}$$

4.5 Commutative limit $\theta \to 0$

In the commutative limit the action reduces to the usual action of gravity coupled to fermions of eq. (2.1). Indeed in virtue of the charge conjugation conditions on V and Ω, the component fields $\tilde V^a$, ω, and $\tilde\omega$ all vanish in the limit $\theta \to 0$ (see the second line of (4.23)), and only the classical spin connection ω^{ab}, vierbein V^a and Dirac fermion ψ survive. Similarly the gauge parameters ε, and $\tilde\varepsilon$ vanish in the commutative limit.

5 Component analysis

We give here the action (4.1) in terms of the component fields V^a, ω^{ab}, $\tilde V^a$, ω, and $\tilde\omega$, and the gauge variations of these fields.

5.1 Action for the component fields

$$\begin{aligned}
S = \int & R^{ab} \wedge_\star (V^c \wedge_\star V^d - \tilde V^c \wedge_\star \tilde V^d)\epsilon_{abcd} \\
& +2i\ R^{ab} \wedge_\star (-V_a \wedge_\star \tilde V_b + \tilde V_a \wedge_\star V_b) \\
& +4i\ r \wedge_\star (V^a \wedge_\star \tilde V_a - \tilde V^a \wedge_\star V_a) \\
& +4i\ \tilde r \wedge_\star (V^a \wedge_\star V_a - \tilde V^a \wedge_\star \tilde V_a) \\
& +Tr[(D\psi \star \bar\psi - \psi \star D\bar\psi)\gamma^d] \wedge_\star [i\epsilon_{abcd}(V^a \wedge_\star V^b \wedge_\star V^c \\
& \qquad -V^a \wedge_\star \tilde V^b \wedge_\star \tilde V^c + \tilde V^a \wedge_\star V^b \wedge_\star \tilde V^c - \tilde V^a \wedge_\star \tilde V^b \wedge_\star V^c) \\
& \qquad +V^a \wedge_\star V_a \wedge_\star \tilde V_d - V^a \wedge_\star \tilde V_a \wedge_\star V_d + \tilde V^a \wedge_\star V_a \wedge_\star V_d - \tilde V^a \wedge_\star \tilde V_a \wedge_\star \tilde V_d
\end{aligned}$$

$$
\begin{aligned}
&+V_d \wedge_\star V^a \wedge_\star \tilde{V}_a - V_d \wedge_\star \tilde{V}^a \wedge_\star V_a + \tilde{V}_d \wedge_\star V^a \wedge_\star V_a - \tilde{V}_d \wedge_\star \tilde{V}^a \wedge_\star \tilde{V}_a \\
&-V^a \wedge_\star V_d \wedge_\star \tilde{V}_a + V^a \wedge_\star \tilde{V}_d \wedge_\star V_a - \tilde{V}^a \wedge_\star V_d \wedge_\star V_a + \tilde{V}^a \wedge_\star \tilde{V}_d \wedge_\star \tilde{V}_a] \\
+&Tr[(D\psi \star \bar{\psi} - \psi \star D\bar{\psi})\gamma^d \gamma_5] \wedge_\star [i\epsilon_{abcd}(V^a \wedge_\star V^b \wedge_\star \tilde{V}^c \\
&-V^a \wedge_\star \tilde{V}^b \wedge_\star V^c + \tilde{V}^a \wedge_\star V^b \wedge_\star V^c - \tilde{V}^a \wedge_\star \tilde{V}^b \wedge_\star \tilde{V}^c) \\
&+V^a \wedge_\star V_a \wedge_\star V_d - V^a \wedge_\star \tilde{V}_a \wedge_\star \tilde{V}_d + \tilde{V}^a \wedge_\star V_a \wedge_\star \tilde{V}_d - \tilde{V}^a \wedge_\star \tilde{V}_a \wedge_\star V_d \\
&+V_d \wedge_\star V^a \wedge_\star V_a - V_d \wedge_\star \tilde{V}^a \wedge_\star \tilde{V}_a + \tilde{V}_d \wedge_\star V^a \wedge_\star \tilde{V}_a - \tilde{V}_d \wedge_\star \tilde{V}^a \wedge_\star V_a \\
&-V^a \wedge_\star V_d \wedge_\star V_a + V^a \wedge_\star \tilde{V}_d \wedge_\star \tilde{V}_a - \tilde{V}^a \wedge_\star V_d \wedge_\star \tilde{V}_a + \tilde{V}^a \wedge_\star \tilde{V}_d \wedge_\star V_a] \qquad (5.1)
\end{aligned}
$$

with

$$
\begin{aligned}
R^{ab} &= d\omega^{ab} - \frac{1}{2}\omega^a{}_c \wedge_\star \omega^{cb} + \frac{1}{2}\omega^b{}_c \wedge_\star \omega^{ca} - \frac{i}{4}(\omega^{ab} \wedge_\star \omega + \omega \wedge_\star \omega^{ab}) - \\
&\quad -\frac{i}{8}\varepsilon^{ab}{}_{cd}(\omega^{cd} \wedge_\star \tilde{\omega} + \tilde{\omega} \wedge_\star \omega^{cd}) \qquad (5.2) \\
r &= d\omega + \frac{1}{8}\omega^{ab} \wedge_\star \omega_{ab} + \omega \wedge_\star \omega - \tilde{\omega} \wedge_\star \tilde{\omega} \\
\tilde{r} &= d\tilde{\omega} - i(\omega \wedge_\star \tilde{\omega} + \tilde{\omega} \wedge_\star \omega) + \frac{i}{16}\varepsilon_{abcd}\omega^{ab} \wedge_\star \omega^{cd} \qquad (5.3)
\end{aligned}
$$

5.2 Gauge variations

$$
\begin{aligned}
\delta_\epsilon V^a &= \frac{1}{2}(\varepsilon^a{}_b \star V^b + V^b \star \varepsilon^a{}_b) + \frac{i}{4}\varepsilon^a{}_{bcd}(\tilde{V}^b \star \varepsilon^{cd} - \varepsilon^{cd} \star \tilde{V}^b) \\
&\quad +\varepsilon \star V^a - V^a \star \varepsilon - \tilde{\varepsilon} \star \tilde{V}^a - \tilde{V}^a \star \tilde{\varepsilon} \qquad (5.4) \\
\delta_\epsilon \tilde{V}^a &= \frac{1}{2}(\varepsilon^a{}_b \star \tilde{V}^b + \tilde{V}^b \star \varepsilon^a{}_b) + \frac{i}{4}\varepsilon^a{}_{bcd}(V^b \star \varepsilon^{cd} - \varepsilon^{cd} \star V^b) \\
&\quad +\varepsilon \star \tilde{V}^a - \tilde{V}^a \star \varepsilon - \tilde{\varepsilon} \star V^a - V^a \star \tilde{\varepsilon} \qquad (5.5) \\
\delta_\epsilon \omega^{ab} &= \frac{1}{2}(\varepsilon^a{}_c \star \omega^{cb} - \varepsilon^b{}_c \star \omega^{ca} + \omega^{cb} \star \varepsilon^a{}_c - \omega^{ca} \star \varepsilon^b{}_c) \\
&\quad +\frac{1}{4}(\varepsilon^{ab} \star \omega - \omega \star \varepsilon^{ab}) + \frac{i}{8}\varepsilon^{ab}{}_{cd}(\varepsilon^{cd} \star \tilde{\omega} - \tilde{\omega} \star \varepsilon^{cd}) \qquad (5.6) \\
&\quad +\frac{1}{4}(\varepsilon \star \omega^{ab} - \omega^{ab} \star \varepsilon) + \frac{i}{8}\varepsilon^{ab}{}_{cd}(\tilde{\varepsilon} \star \omega^{cd} - \omega^{cd} \star \tilde{\varepsilon}) \qquad (5.7) \\
\delta_\epsilon \omega &= \frac{1}{8}(\omega^{ab} \star \varepsilon_{ab} - \varepsilon_{ab} \star \omega^{ab}) + \varepsilon \star \omega - \omega \star \varepsilon + \tilde{\varepsilon} \star \tilde{\omega} - \tilde{\omega} \star \tilde{\varepsilon} \qquad (5.8) \\
\delta_\epsilon \tilde{\omega} &= \frac{i}{16}\varepsilon_{abcd}(\omega^{ab} \star \varepsilon^{cd} - \varepsilon^{cd} \star \omega^{ab}) + \varepsilon \star \tilde{\omega} - \tilde{\omega} \star \varepsilon + \tilde{\varepsilon} \star \omega - \omega \star \tilde{\varepsilon} \qquad (5.9)
\end{aligned}
$$

6 Noncommutative Mac-Dowell Mansouri gravity

6.1 Action and symmetries

As already discussed in [18], the noncommutative generalization of the Mac-Dowell Mansouri action [19] reads:

$$
S = i\int Tr[R \wedge_\star R\gamma_5] \qquad (6.1)
$$

with

$$R = d\Omega - \Omega \wedge_\star \Omega \tag{6.2}$$

and

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} + i\omega 1 + \tilde{\omega}\gamma_5 + iV^a\gamma_a + i\tilde{V}^a\gamma_a\gamma_5 \tag{6.3}$$

The $GL(2,C)$ $\star$-gauge variations act as:

$$\delta_\epsilon \Omega = d\epsilon - \Omega \star \epsilon + \epsilon \star \Omega \tag{6.4}$$

with

$$\epsilon = \frac{1}{4}\varepsilon^{ab}\gamma_{ab} + i\varepsilon 1 + \tilde{\varepsilon}\gamma_5 \tag{6.5}$$

so that

$$\delta_\epsilon R = -R \star \epsilon + \epsilon \star R \tag{6.6}$$

The invariance of the action (6.1) under $\star$-gauge transformations is easily checked, taking into account the transformation of R, the cyclicity of the trace Tr, the graded cyclicity of the integral and the fact that ϵ still commutes with γ_5.

6.2 Hermiticity and charge conjugation

Hermiticity conditions can again be imposed on Ω and on the gauge parameter ϵ:

$$-\gamma_0\Omega\gamma_0 = \Omega^\dagger, \qquad -\gamma_0\epsilon\gamma_0 = \epsilon^\dagger \tag{6.7}$$

These conditions are consistent with the gauge variations, and can be used to check that the action (6.1) is real. Again the hermiticity conditions imply that the component fields V^a, $\tilde{V}^a$, ω^{ab}, ω, $\tilde{\omega}$, and the component gauge parameters ε^{ab}, ε, $\tilde{\varepsilon}$ are real.

The charge conjugation conditions are again

$$C\Omega_\theta(x)C = \Omega(x)^T_{-\theta}, \qquad C\varepsilon_\theta(x)C = \varepsilon(x)^T_{-\theta} \tag{6.8}$$

These conditions are consistent with the $\star$-gauge transformations.
For the component fields and gauge parameters the charge conjugation conditions imply the same relations (4.23), (4.25) as in section 4.

6.3 Commutative limit $\theta \to 0$

In the commutative limit the action reduces to the usual action of Mac Dowell-Mansouri gravity. Indeed the charge conjugation conditions on Ω ensure that the component fields $\tilde{V}^a$, ω, and $\tilde{\omega}$ all vanish in the limit $\theta \to 0$, and only the classical spin connection ω^{ab}, vierbein V^a survive. Moreover the gauge parameters ε and $\tilde{\varepsilon}$ vanish in the limit because of the charge conjugation condition on ϵ , and only the parameter ε^{ab} corresponding to Lorentz symmetry survives.

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7 Conclusions

We have constructed a geometric noncommutative action of first-order gravity coupled to fermions, invariant under $\star$-diffeomorphisms and $GL(2,C)$ $\star$-gauge transformations. The commutative limit reproduces the usual action with no extra fields, and the $\star$-invariance reduces to ordinary Lorentz invariance. A charge conjugation condition, consistent with the $\star$-symmetries, is imposed on the noncommutative vielbein and connection, and takes into account their θ-dependence. This condition allows to recover the usual commutative limit. Finally, using the same tools of twisted differential geometry, we find the noncommutative extension of the Mac-Dowell Mansouri action.

A Gamma matrices in $D = 4$

We summarize in this appendix our gamma conventions in $D = 4$.

$$\eta_{ab} = (1,-1,-1,-1), \quad \{\gamma_a,\gamma_b\} = 2\eta_{ab}, \quad [\gamma_a,\gamma_b] = 2\gamma_{ab}, \tag{A.1}$$
$$\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5\gamma_5 = 1, \quad \varepsilon_{0123} = -\varepsilon^{0123} = 1, \tag{A.2}$$
$$\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0, \quad \gamma_5^\dagger = \gamma_5 \tag{A.3}$$
$$\gamma_a^T = -C\gamma_a C^{-1}, \quad \gamma_5^T = C\gamma_5 C^{-1}, \quad C^2 = -1, \quad C^T = -C \tag{A.4}$$

A.1 Useful identities

$$\gamma_a\gamma_b = \gamma_{ab} + \eta_{ab} \tag{A.5}$$
$$\gamma_{ab}\gamma_5 = \frac{i}{2}\epsilon_{abcd}\gamma^{cd} \tag{A.6}$$
$$\gamma_{ab}\gamma_c = \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \tag{A.7}$$
$$\gamma_c\gamma_{ab} = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a - i\varepsilon_{abcd}\gamma_5\gamma^d \tag{A.8}$$
$$\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \tag{A.9}$$
$$\gamma^{ab}\gamma_{cd} = -i\varepsilon^{ab}{}_{cd}\gamma_5 - 4\delta^{[a}_{[c}\gamma^{b]}{}_{d]} - 2\delta^{ab}_{cd} \tag{A.10}$$

where $\delta^{ab}_{cd} = \frac{1}{2}(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c)$, and index antisymmetrizations in square brackets have weight 1.

A.2 Charge conjugation and Majorana condition

$$\text{Dirac conjugate} \quad \bar\psi \equiv \psi^\dagger\gamma_0 \tag{A.11}$$
$$\text{Charge conjugate spinor} \quad \psi^c = C(\bar\psi)^T \tag{A.12}$$
$$\text{Majorana spinor} \quad \psi^c = \psi \;\Rightarrow \bar\psi = \psi^T C \tag{A.13}$$

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